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# Commuting Boundedly Decomposable Perturbations

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The author proves that perturbation of a decomposable operator by a commuting boundedly decomposable operator is decomposable, thus generalizing a theorem of C. Apostol. It is also proved that the sum of a regular $A$-spectral operator and a commuting boundedly decomposable operator is strongly decomposable, while the sum of two commuting boundedly decomposable operators is generalized spectral. Examples show that boundedly decomposable operators form a proper intermediate class between spectral and generalized spectral operators; thus the generalization of Apostol's theorem is not trivial. 

## 1. INTRODUCTION

C. Apostol has proved [2] that the perturbation of a decomposable operator by a commuting spectral operator (Dunford type) is again decomposable. In this paper we generalize this result by showing that the sum of a decomposable and a commuting boundedly decomposable operator is also decomposable. An example proves this generalization to be nontrivial.

Boundedly decomposable operators were introduced by Evans [8], who gave characterizations and structure theorems [8, Theorems 2, 3], and he showed that such operators are closely related to spectral operators and generalized spectral operators. Below we show that the *class* of boundedly decomposable operators lies strictly between the former two.

## 2. PERTURBATIONS OF DECOMPOSABLE OPERATORS

Our first result is due to Apostol [2, p. 1498], but we give a simplified proof for later reference. First recall that by [9] a bounded linear operator $T$ on the complex Banach space $X$ is decomposable if for each finite open cover $\{G_1, G_2, ..., G_n\}$ of the complex plane $\mathbb{C}$ there corresponds a system of closed $T$-invariant subspaces $M_1, ..., M_n$ such that $X = M_1 + M_2 + \cdots + M_n$ and $\sigma(T|M) \subset G_j (j = 1, 2, ..., n)$.



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THEOREM 1. *Let $T$ be decomposable and let $S$ be a spectral operator commuting with $T$. Then $TS$ and $T+S$ are also decomposable.*

*Proof.* First suppose that $S$ is scalar-type, and let $\varepsilon > 0$. Let $E$ be the resolution of the identity for $S$, let $\alpha = \{\beta_j\}$ be a finite Borel partition of $\sigma(S)$ (or $\mathbb{C}$) and let $\lambda_j \in \beta_j$ such that (see [5])

$$\left\| S - \sum \lambda_j E(\beta_j) \right\| < \varepsilon \|T\|^{-1}.$$

If we put $E_j = E(\beta_j)$ (each $j$) then

$$\left\| TS - \sum \lambda_j TE_j \right\| < \varepsilon. \tag{1}$$

Without loss of generality [10], let $\{G_1, G_2\}$ be an open cover of $\sigma(TS)$ where $G_i$ are relatively compact. For each $\alpha$ as above let $U_\alpha = \sum \lambda_j TE_j$. If we put $T_j = T_j | E_j X$, then $U_\alpha = \bigoplus \lambda_j T_j$ and $U_\alpha$ is decomposable by [3, Props. 1.8 and 1.9]. It is then evident that if we choose a second cover $\{H_1, H_2\}$ of $\sigma(TS)$ with $H_i \subset \bar{H}_i \subset G_i (i = 1, 2)$ we may find spectral maximal spaces $M_1$, $M_2$ of $U_\alpha$ such that $X = M_1 + M_2$ and $\sigma(U_\alpha | M_i) \subset H_i$ (see [3, p. 30]). Then $M_i$ are invariant under $TS$, so by (1) and the continuity of the Hausdorff metric for spectra of commuting operators, we may suppose $\sigma(TS|M_i) \subset G_i$ $(i = 1, 2)$. By the remark above $TS$ is decomposable.

If $S$ is spectral then $S = A + Q$ where $A$ is scalar-type, $Q$ is quasinilpotent and commutes with $A$ and both these commute with $T$. Hence $TS = TA + TQ$ is decomposable by the first part of the proof and [3, Th 2.2.1].

Decomposability and spectrality are both invariant under the analytic functional calculus [2, 4], hence for $\lambda \in \rho(-S)$

$$T + S = (\lambda + S)[(T - \lambda) R(\lambda; -S) + I] \tag{2}$$

is decomposable by the previous paragraph. The proof is thus complete.

Our principal aim in this section is to extend Theorem 1 to the case where $S$ is boundedly decomposable in the following sense of Evans [8].

DEFINITION 1. Let $T$ be decomposable on $X$. Then $T$ is boundedly decomposable if there exists $m > 0$ such that for each open cover $\{G_1, G_2\}$ of $\sigma(T)$ and each $x \in X$ there are vectors $x_i \in X$ with $x = x_1 + x_2$, $\sigma(x_i, T) \subset G_i$, and $\max\{\|x_1\|, \|x_2\|\} \leqslant m\|x\|$. (For details on local spectra $\sigma(x, T)$ see [3, p. 1]).

We shall use the following structure theorems from [8]. The simplified version of Theorem 2 given here follows from [8, Prop. 5; 7, p. 70].

THEOREM 2. *For an operator $T$ on $X$ the following are equivalent.*

(i) *$T$ is boundedly decomposable;*

(ii) *$T^*$ is prespectral of class $X$;*

(iii) *$T^*$ is boundedly decomposable.*

THEOREM 3. *Let $T$ be boundedly decomposable.*
*Then $T = S + Q$ where*

(i) *$S$ is $C$-scalar where $C$ is the set of $f: \sigma(S) \to \mathbb{C}$ continuous on $\sigma(S)$;*

(ii) *$f(S)$, $Q$ are in the bicommutant of $T$;*

(iii) *$Q$ is quasinilpotent;*

(iv) *this decomposition is unique in* (i)–(iii).

(*For details on $C$-scalar operators, see either Definition* 2 *below or* [3, Chap. 3].)

THEOREM 4. *Every spectral operator is boundedly decomposable but not conversely.*

*Proof.* Let $T$ be spectral with spectral measure $E$ and uniform bound $m$. Then $T$ is decomposable. For an open cover $\{G_1, G_2\}$ of $\sigma(T)$ let $\{\beta_1, \beta_2\}$ be a Borel partition of $\sigma(T)$ with $\bar{\beta}_i \subset G_i$; for $x \in X$, let $x_i = E(\beta_i)\, x$. Definition 1 is now easily verified.

To see the failure of the converse, let $X = l^1$ and define $T$ on $l^1$ by $T(x_n) = (n^{-1} x_n)$. By [6, p. 2079] $T$ is spectral but $T^*$ is not spectral. On the other hand, $T^*$ is boundedly decomposable by the first part of the proof and Theorem 2.

The last example shows the following generalization of Theorem 1 to be nontrivial.

THEOREM 5. *Let $T$ be decomposable, and let $S$ be a commuting boundedly decomposable operator. Then $TS$ and $T + S$ decomposable.*

*Proof.* Since the analytic (Riesz–Dunford) functional calculus preserves decomposability [3, p. 37] and scalar translations clearly preserve bounded decomposability, by relation (2) above, it suffices to prove $TS$ is decomposable. This will follow if we can prove that $T^*S^*$ is decomposable [7, Th. 9.6].

First let $S$ be $C$-scalar, i.e., suppose $Q = 0$ in Theorem 3. By Theorem 2 $S^*$ is prespectral (scalar-type) of class $X$, hence $T^*S^*$ can be proved decomposable by the method of Theorem 1 once we prove that $T^*$ commutes with the spectral measure $E$ of $S^*$. By Theorem 3, $T^*f(S^*) = f(S^*)\, T^*$ for each continuous $f$. Now fix $x \in X$, $u \in X^*$. For each Borel set $\beta$ define complex-valued measures

$$\begin{aligned} \mu_1(\beta) &= \langle x, E(\beta)\, T^*u \rangle \\ \mu_2(\beta) &= \langle Tx, E(\beta)\, u \rangle. \end{aligned} \tag{3}$$

By [4, Th. 5.8, p. 122] and the fact that $T^*$ and $f(S^*)$ commute

$$\int f(\lambda)\, d\mu_1(\lambda) = \langle x, f(S^*)\, T^*u \rangle = \langle Tx, f(S^*)\, u \rangle$$

$$= \int f(\lambda)\, d\mu_2(\lambda).$$

Since $\mu_1$ and $\mu_2$ are regular with supports clearly in $\sigma(S^*)$, $\mu_1 = \mu_2$ by the Riesz representation theorem. Hence (3) implies $\langle x, E(\beta)\, T^*u \rangle = \langle Tx, E(\beta)\, u \rangle$. It follows that $T^*E(\beta) = E(\beta)\, T^*$ for each Borel set $\beta$, hence $T^*S^*$ is decomposable and thus $TS$ also.

Finally if $S$ is an arbitrary boundedly decomposable operator commuting with $T$ with decomposition $S = S_1 + Q$ given by Theorem 3, then by (ii) of that theorem $T$ commutes with both $S_1$ and $Q$. But $TS_1$ is decomposable by the first part of the proof and $TQ$ is quasinilpotent and commutes with $TS_1$. Hence $TS$ is decomposable by [3, Th. 2.2.1, p. 40]. Now the proof is complete.

## 3. Special Subclasses

Specializing $T$ in Theorem 5 above allows us to obtain stronger conclusions. For further details on the following definition, see [3, pp. 59, 66, 78].

Definition 2. Let $A$ be a normal "admissible" algebra of functions $f: \mathbb{C} \to \mathbb{C}$. An algebra homomorphism $U: A \to L(X)$ ( = all operators on $X$) is called an *$A$-spectral function.* If $T = U(id)$ (where $id(\lambda) = \lambda$) then $T$ is *$A$-scalar*, and $T$ is called *$A$-spectral* if $T = S + Q$ where $S$ is $A$-scalar, $Q$ is quasinilpotent and $QU(f) = U(f)\, Q$ for all $f \in A$. Finally we say that $T$ is *regular* if $U(f)$ is in the bicommutant of $T$ for all $f \in A$.

*Remark.* Theorem 3 shows that boundedly decomposable operators are regular $C$-spectral.

Definition 3. A decomposable operator $T$ is *strongly* decomposable if $T|M$ is decomposable for each spectral maximal space $M = X(T, F)$, $F$ closed (see [2, p. 1489]).

THEOREM 6. *Let $T$ be a regular $A$-scalar operator and let $S$ be $C$-scalar commuting with $T$. Then $T+S$ is strongly decomposable.*

*Proof.* By Theorem 5 $T+S$ is decomposable. Let $R=T+S$. By [2, p. 1489] we need to prove that the restriction $R \mid X(R, F)$ is decomposable for each closed set $F$ in $\mathbb{C}$ (see [3, p. 31]). By Theorem 3 $R$ commutes with $f(S)$ for each continuous $f$. Hence $X(R, F)$ is $f(S)$-invariant and so the map $f \to f(S) \mid X(R, F)$ is a continuous functional calculus [8]. Thus $S \mid X(R, F)$ is $C$-scalar. Let $U: A \to L(X)$ be an $A$-spectral function for $T$. Since $T$ is regular $U(g)\, R = RU(g)$ for all $g \in A$, so the map $g \to U(g) \mid X(R, F)$ is an $A$-spectral function for $T \mid X(R, F)$. By Theorem 5, $R \mid X(R, F)$ is decomposable, hence $R=T+S$ is strongly decomposable.

COROLLARY 1. *If $T$ is a regular $A$-spectral operator and $S$ is a commuting boundedly decomposable operator, then $T+S$ is strongly decomposable.*

*Proof.* Let $T$ be a regular $A$-spectral operator such that $T=V+Q$ where $V$ is $A$-scalar and $Q$ is quasinilpotent and commutes with $U(f)$ $(f \in A)$. Let $S$ be boundedly decomposable with $S=W+R$ where $W$ is $C$-scalar and $R$ is quasinilpotent as in Theorem 3. Denote the continuous functional calculus of $W$ by $g(W)$, $g \in C$. Since $TS=ST$ we have $Tg(W)=g(W)\, T$ for $g \in C$, hence $TR=RT$. Regularity of $T$ implies $RU(f)=U(f)\, R$ $(f \in A)$, so $g(W)\, U(f)=U(f)\, g(W)$ for all $f \in A$, $g \in C$. Next $g(W)\, Q=Qg(W)$, so finally $QR=RQ$.

Just as in the proof of Theorem 6 we have $V+W$ strongly decomposable. Since $R+Q$ is quasinilpotent and commutes with $V+W$, it follows from [3, Th. 2.2.1] as before that $T+S$ is strongly decomposable.

Since every boundedly decomposable operator is regular $C$-spectral by Theorem 3, the next corollary is immediate from Corollary 1.

COROLLARY 2. *The sum of two commuting boundedly decomposable operators is strongly decomposable.*

A consequence of the next theorem is a much stronger conclusion than Corollary 2. To state this result we need to recall the notion of generalized spectral operator.

DEFINITION 4. Let $T$ be $A$-scalar (-spectral) where $A=C^{\infty}$, the algebra of infinitely differentiable functions on $\mathbb{C}$. If the $C^{\infty}$-spectral function $f \to U(f)$ is continuous in the topology of uniform convergence on compact sets in $\mathbb{C}$, then $T$ is called *generalized scalar* (*spectral*). In this case $f \to U(f)$ is called spectral distribution of $T$.

THEOREM 7. *Every boundedly decomposable operator is generalized spectral.*

*Proof.* Let $T$ be boundedly decomposable. By Theorem 3 we may suppose that $T$ is $C$-scalar, and we must prove that $T$ is generalized scalar. By Theorem 2, $T^*$ is prespectral scalar-type of class $X$. Let $E$ be the spectral measure of $T^*$. By [4, Th. 5.21]

$$(f(T))^* = \int_{\sigma(T^*)} f(\lambda)\, dE(\lambda) \qquad (f \in C(\sigma(T^*))).$$

From the uniform boundedness of $E$ and the Taylor expansion for $f \in C^\infty$ we infer that the map $U: C^\infty \to L(X^*)$ given by $U(f) = (f(T))^*$ is a spectral distribution of $T$ in the sense of Definition 4. Thus $f \to f(T)$ $(f \in C^\infty)$ is also a spectral distribution of $T$, so $T$ is generalized scalar by Definition 4.

THEOREM 8. *The sum (product) of two commuting boundedly decomposable operators is generalized spectral.*

*Proof.* Let $T_1$ and $T_2$ be commuting boundedly decomposable operators with decompositions (Theorem 3) $T_i = S_i + Q_i$ $(i = 1, 2)$ into $C$-scalar and quasinilpotent parts. Since $T_i$ are regular by Theorem 3, $f(S_1)$ and $g(S_2)$ commute for $f, g \in C^\infty$, and similarly $Q_1$ and $Q_2$ commute with these and with each other. The result now follows from Theorem 7 and [3, Cor. 4.3.9].

We close with an example that shows that the converse of Theorem 7 is false.

EXAMPLE. A generalized scalar operator that is not boundedly decomposable. Albrecht [1, Cor. 2.8] gives an example of a nonregular generalized scalar operator. But every boundedly decomposable operator is regular, hence Albrecht's example cannot be boundedly decomposable.

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